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A DESARGUESIAN THEOREM FOR ALGEBRAIC COMBINATORIAL GEOMETRIES

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The points of an algebraic combinatorial geometry are equivalence classes of transcendentals over a field k; two transcendentals represent the same point when they are algebraically dependent over k. The points of an algebraically closed field of transcendence degree two (three) over k are the lines (resp. planes) of the geometry.

We give a necessary and sufficient condition for two coplanar lines to meet in a point (Theorem 1) and prove the converse of Desargues' theorem for these geometries (Theorem 2). A corollary: the "non-Desargues" matroid is non-algebraic.

The proofs depend on five properties (or postulates). The fifth of these is a deep property first proved by Ingleton and Main [3] in their paper showing that the Vámos matroid is non-algebraic.

We consider geometries of points, lines and planes (point-sets) with the following properties:

P1. Any two distinct points p and q determine a unique line $\overline{p, q}$, with $p, q, \in \overline{p, q}$.
P2. Any three distinct points p, q, r determine a unique plane $\overline{p, q, r}$ with $p, q, r \in \overline{p, q, r}$.
P3. If two points p and q belong to a plane π, then $\overline{p, q} \subseteq \pi$.

P4. For any plane π_1 and line $l \subseteq \pi_1$ there is another plane π_2 such that $\pi_1 \cap \pi_2 = l$. Also there is an isomorphism $\varphi\colon \pi_1 \to \pi_2$ which preserves incidences between points and lines (i.e. $p \in l$ implies $\varphi(p) \in \varphi(l)$) which extends the identic map id: $l \to l$.
P5. Let $\overline{a_1, b_1}, \overline{a_2, b_2}, \overline{a_3, b_3}$ be three distinct lines which are pairwise coplanar, but not all three coplanar. Then the lines meet in a point.

We have a class of geometries with this properties in mind. Consider an algebraically closed field K of transcendence degree 4 over a field k. A point is an equivalence class of transcendentals in K. Two transcendentals represent the same point when they are algebraically dependent over k. We denote the point by $\bar{x}$ when x is a transcendental with $x \in \bar{x}$. If $x, y \in K$ are algebraically independent over k let $\overline{k(x, y)}$ denote the algebraic closure of $k(x, y)$ in K. The points of $\overline{k(x, y)}$ is the line $\overline{x, y}$ determined by $\bar{x}$ and $\bar{y}$. The plane $\overline{x, y, z}$ determined by three points $\bar{x}, \bar{y}, \bar{z}$ is defined similarly. The point, lines and planes are flats of a combinatorial geometry in

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the sense of [2]. Properties P1—P3 are satisfied by any combinatorial geometry. For P4 see e.g. Corollary 4.2.8 in [1]: if k is a field, then every two algebraically closed extension fields of k having the same transcendence degree over k are k-isomorphic. Property P5 is deeper.

There is a proof in [3] by Ingleton and Main as a part of their proof that the Vámos matroid is non-algebraic. Lovász has found another proof of P5 [6].

Note that we do not assume that coplanar lines intersect in a point.

Theorem 1. *Necessary and sufficient for two coplanar lines* $\overline{a_1, b_1}$ *and* $\overline{a_2, b_2}$ *to intersect in a point is that* $\bar{a}_2, \bar{b}_2, \overline{\varphi(a_2)}, \overline{\varphi(b_2)}$ *are coplanar points, when* $l=\overline{a_1, b_1}, \pi_1$ *is the plane* $\overline{a_1, b_1, a_2, b_2}$ *and* φ *is an isomorphism* $\varphi\colon \pi_1 \to \pi_2$ *as in* P4.

Proof. The necessity follows by P1—P3. The sufficiency of the condition follows by P5. ▌

Example. The lines $\overline{x, y}$ and $\overline{xz+y, z}$ in the $\overline{x, y, z}$-plane do not meet. Let $\pi_1 = \overline{x, y, z}$ and $\pi_2 = \overline{x, y, u}$, where x, y, z, u are algebraically independent transcendentals over k. Let $\varphi(\bar{x})=\bar{x}$, $\varphi(\bar{y})=\bar{y}$, $\varphi(\bar{z})=\bar{u}$, where $\varphi\colon \pi_1 \to \pi_2$ is an isomorphism. We need only prove that $\bar{z}, \bar{u}, \overline{xz+y}, \overline{xu+y}$ are not coplanar. Assume that these are four points in a plane π. Write $xz+y=a$, $xu+y=b$. Then it follows $x=(a-b)/(z-u)$. Hence $\bar{x}, \bar{y}, \bar{z}, \bar{u} \in \pi$, a contradiction. Therefore the lines do not meet. A similar example was in [7] by S. Mac Lane. ▌

Definition. Two point-triples $(\bar{a}_1, \bar{a}_2, \bar{a}_3)$ and $(\bar{b}_1, \bar{b}_2, \bar{b}_3)$ are said to be *centrally perspective* when the lines $\overline{a_i, b_i}$ $(i=1, 2, 3)$ meet in a point. The two triples are said to be *axially perspective* if the lines $\overline{a_i, a_j}$ and $\overline{b_i, b_j}$ meet in a point $\bar{c}_{ij}$ $(1 \leq i < j \leq 3)$ and if the points $\bar{c}_{12}, \bar{c}_{13}, \bar{c}_{23}$ are unique, distinct and collinear.

Theorem 2. *Let the point-triples* $(\bar{a}_1, \bar{a}_2, \bar{a}_3)$ *and* $(\bar{b}_1, \bar{b}_2, \bar{b}_3)$ *(6 points) in a plane be axially perspective. Then the two point-triples are centrally perspective.*

Corollary. *The "non-Desargues" matroid* [8, p. 139] *is non-algebraic.* ▌

Proof. Let π_1 denote the plane to which $\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{b}_1, \bar{b}_2, \bar{b}_3$ belong. Let l be the axis of perspectivity. Let π_2 be a plane which intersects π_1 in l according to P4. Let $\varphi(\bar{b}_i) = \bar{d}_i$ $(i=1, 2, 3)$ be images according to P4.

Note that the lines $\overline{d_i, d_j}$, $\overline{a_i, a_j}$ and $\overline{b_i, b_j}$ meet in the point $\bar{c}_{ij}$ on l. The lines are therefore pairwise coplanar. It follows then that the lines $\overline{a_i, d_i}$ $(i=1, 2, 3)$ satisfy the conditions of P5. These lines therefore meet in a point $\bar{r}$. Similarly it follows that the lines $\overline{b_i, d_i}$ $(i=1, 2, 3)$ meet in a point $\bar{s}$. Note that $\bar{r}$ and $\bar{s}$ are distinct points, which do not belong to π_1 or π_2. Also note that the four points $\bar{a}_i, \bar{b}_i, \bar{r}, \bar{s}$ are coplanar, since the lines $\overline{a_i, r}$ and $\overline{b_i, s}$ meet in the point $\bar{d}_i$ $(i=1, 2, 3)$.

It follows that the lines $\overline{a_i, b_i}, \overline{a_j, b_j}, \overline{r, s}$ $(i \neq j)$ satisfy the conditions of P5 and therefore meet in a point $\bar{c}$ which does not depend on (i, j) $1 \leq i < j \leq 3$. It follows that $(\bar{a}_1, \bar{a}_2, \bar{a}_3)$ and $(\bar{b}_1, \bar{b}_2, \bar{b}_3)$ are centrally perspective. ▌

Remark. Theorem 2 is formally the converse of Desargues' theorem in projective geometry. The reason why we do not prove Desargues' theorem is that the sides of two triangles do not always intersect in points in algebraic combinatorial geometries. In projective geometry, where two coplanar lines always meet, Desargues' theorem and its converse are equivalent.

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